Solid State Theory

Integer quantum Hall effect

May 8, 2023

Agenda

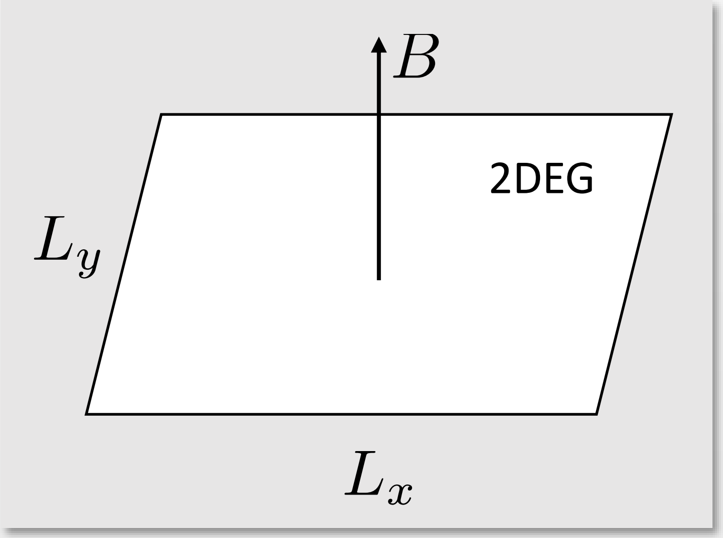
- Integer quantum Hall effect
- Current response of a LL
- Edge states
- Percolation, delocalization, and Hall transition
- Laughlin’s argument
- TKNN number

Integer quantum Hall effect (QHE)

- IQHE $\sigma_{xy} = \nu \frac{e^2}{h}$, filling factor $\nu \in \mathbb{Z}$

$$\rho_{yx} = \frac{1}{\nu}\frac{h}{e^2}$$

- Filling factor $\frac{1}{\nu} = \frac{N_\Phi}{N_e}$



First experiment

Von Klitzing et al.
Phys. Rev. Lett. **45**, 494 (1983).

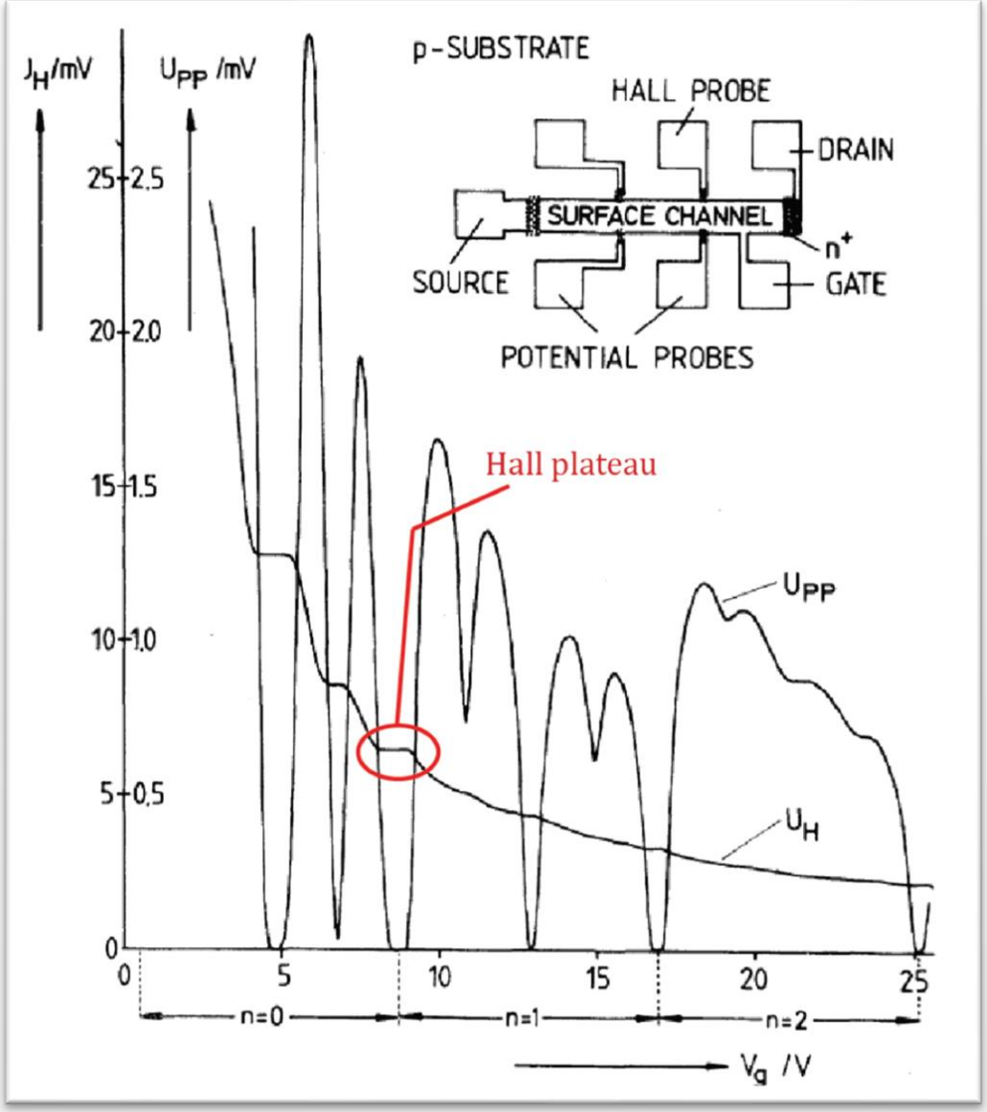
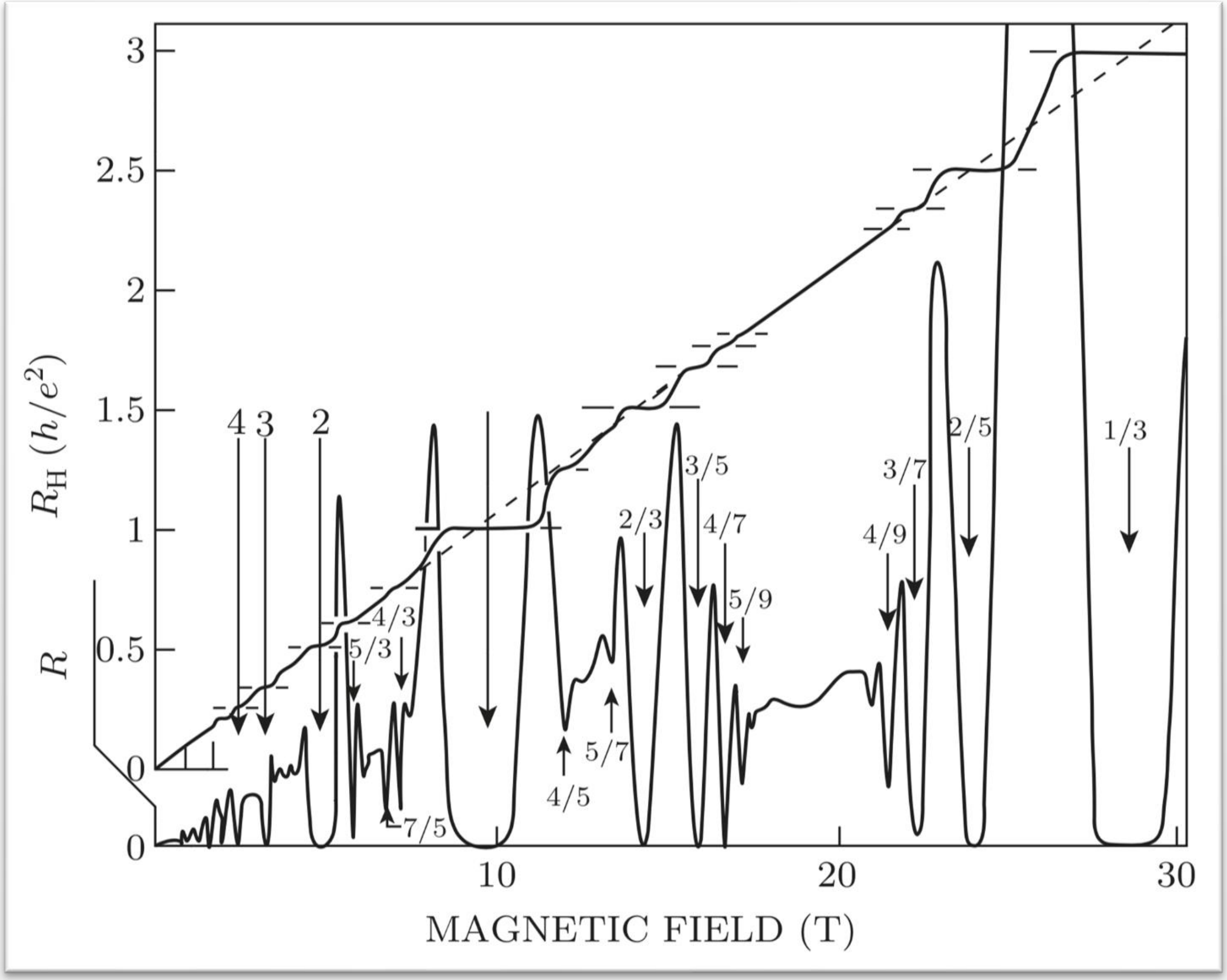


FIG. 1. Recordings of the Hall voltage U_H, and the voltage drop between the potential probes, U_{pp}, as a function of the gate voltage V_g at $T = 1.5$ K. The constant magnetic field (B) is 18 T and the source drain current, I, is 1 μA. The inset shows a top view of the device with a length of $L = 400$ μm, a width of $W = 50$ μm, and a distance between the potential probes of $L_{pp} = 130$ μm.

Zoo of quantum Hall plateaux



H. L. Störmer.
Phys. B: Cond. Matt. **177**, 401(1992)

Disorder is important

- Lorentz boost on a perfect 2DEG $J' = -nev$

Lab frame

$$\boldsymbol{E} = 0$$

$$\boldsymbol{B} = B\hat{z}$$

Moving frame (-v)

$$\boldsymbol{E}' = -\boldsymbol{v} \times \boldsymbol{B}$$

$$\boldsymbol{B}' = \boldsymbol{B}$$

$$\Rightarrow \boldsymbol{E}' = \frac{B}{ne} \boldsymbol{J}' \times \hat{B}$$

$$\Rightarrow \rho = \frac{B}{ne} \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

No quantization!

- Must have disorder to observe QHE

Hamiltonian

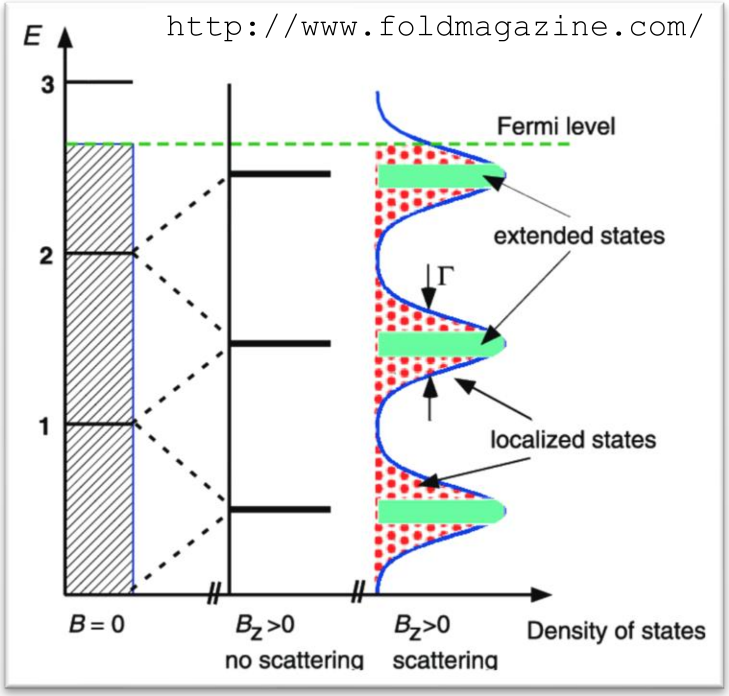
- Landau gauge $a = -Bx\hat{y}$
- Hamiltonian $H = \frac{1}{2m}\left[p_x^2 + (p_y - eBx)^2\right]$ Transl. inv. in y
- Wavefunction $\Psi(x,y) = e^{iky}\psi_k(x)$

$$H\Psi(x,y) = e^{iky}\frac{1}{2m}[p_x^2 + (\hbar k - eBx)^2]\psi_k(x)$$

$$\Rightarrow H_k = \frac{1}{2m}[p_x^2 + \tfrac{1}{2}m\omega_c^2(x - x_k)^2]$$

Cyclotron frequency $\omega_c = \frac{eB}{m}$ Magnetic length $l_B = \sqrt{\frac{\hbar}{eB}}$ Guiding center $x_k = l_B^2 k$

Landau levels


http://www.foldmagazine.com/

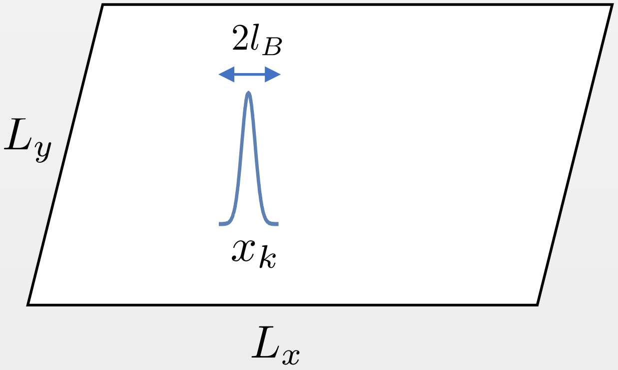
- Energy levels $\varepsilon_{nk} = (n + \frac{1}{2})\hbar\omega_c$

Indep. of k → high degree of degeneracy

- Wavefunctions

Hermite polynomial

$$\psi_{nk}(\boldsymbol{r}) = \frac{1}{\sqrt{A}} e^{iky} H_n((x - x_k)/l_B) \exp\left[-\frac{(x - x_k)^2}{2l_B^2}\right]$$



$$x_k \in [0, L_x] \Rightarrow k \in [0, L_x/l_B^2]$$

- ➢ States on the left and right edges have very different k values
- ➢ Number of states in a LL = N_Φ

$$\nu = N_e/N_\Phi$$

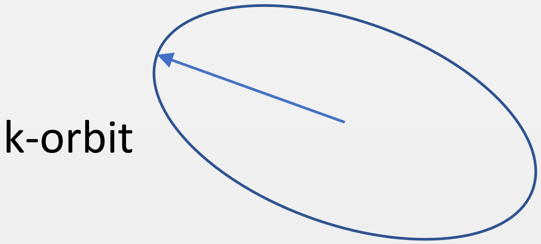
Quantum dynamics of LLs

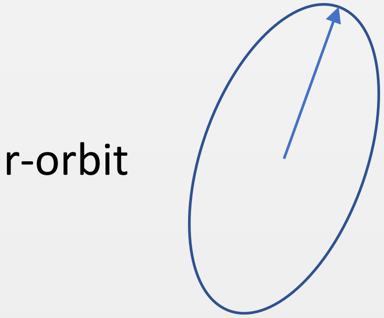
- Arbitrary wavefunction

$$\Psi(\boldsymbol{r},t)=\frac{L_y}{2\pi}\sum_n\int dk a_n(k)\psi_{nk}(\boldsymbol{r})e^{-\mathrm{i}\left(n+\frac{1}{2}\right)\omega_c t}$$

$$\Rightarrow \text{periodic motion} : \Psi(\boldsymbol{r},t+2\pi/\omega_c)=\Psi(\boldsymbol{r},t)$$

Correponding to the classical cyclotron orbits





Current response without E-field

- Lowest LL (LLL): $r_k(\boldsymbol{r}) = \frac{1}{\sqrt{\pi^{1/2}L_y l_B}} e^{iky} e^{-\frac{1}{2\ell^2}\left(x-kl_B^2\right)^2}$
- Current density for a given k

$$\boldsymbol{j}(\boldsymbol{r}) = -\frac{e}{m_{\mathrm{e}}}\psi_k^*(\boldsymbol{r})\left[-i\hbar\nabla + e\boldsymbol{a}(\boldsymbol{r})\right]\psi_k(\boldsymbol{r})$$

- y-component of current

$$I_y = -\frac{e}{m_{\mathrm{e}}\pi^{1/2}l_B}\frac{1}{L_y}\int_{-\infty}^{\infty} dx e^{-\frac{1}{2l_B^2}(x-x_k)^2}\left(\hbar k - eBx\right) e^{-\frac{1}{2l_B^2}(x-x_k)^2}$$

$$= \frac{e\omega_{\mathrm{c}}}{\pi^{1/2}\ell}\frac{1}{L_y}\int_{-\infty}^{\infty} dx e^{-\frac{1}{\ell^2}(x-x_k)^2}\left(x-x_k\right) = 0$$

- ➢ Flux from planewave in y-direnction gets canceled by the vector potnetial contribution!
- ➢ Group velocity vanishes $v_y = \frac{1}{\hbar}\frac{\partial\varepsilon_{kn}}{\partial k} = 0$

Effect of an E-field

- Adding E-field: $V(\boldsymbol{r}) = +eEx$

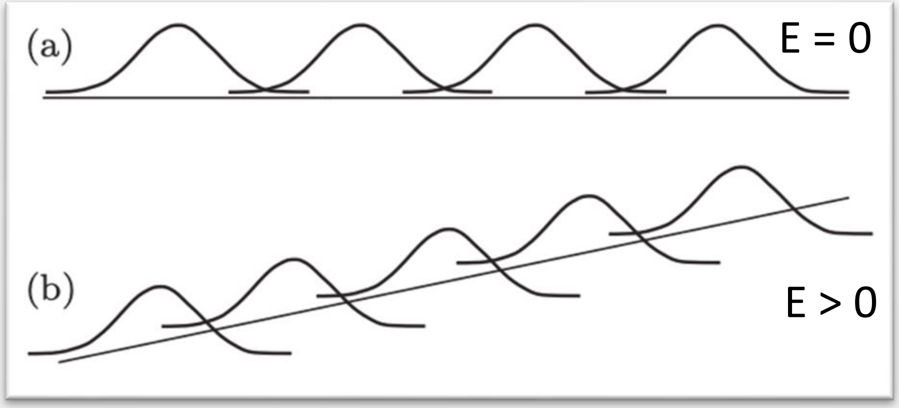
$$H_k = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_c^2\left(x - x_k\right)^2 + eEx$$
$$= \frac{p_x^2}{2m} + \frac{1}{2}m\omega_c^2\left(x - x_k'\right)^2 + eEx_k' + \frac{1}{2}mv_d^2$$

New guiding center $x_k' = x_k - \frac{eE}{m\omega_c}$

Drift velocity $v_d = \frac{E}{B}$

- LLs in electric field

$$\varepsilon_{nk} = (n + \tfrac{1}{2})\hbar\omega_c + \underset{\substack{\uparrow \\ \text{Potential energy}}}{eEx_k'} + \underset{\substack{\uparrow \\ \text{Kinetic energy}}}{\frac{1}{2}mv_d^2}$$



Current response

With electric field: $x'_k = x_k - \frac{eE}{m\omega_c}$ $v_d = \frac{E}{B}$

current

$$
\begin{aligned}
I_y(k) &= \frac{e\omega_c}{\pi^{1/2} l_B} \frac{1}{L_y} \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{1}{l_B^2}\left(x-x'_k\right)^2} (x - x_k) \\
&= \frac{e\omega_c}{\pi^{1/2} l_B} \frac{1}{L_y} \sqrt{\pi} l_B \left(x'_k - x_k\right) \\
&= \frac{1}{L_y} e\omega_c \left(-\frac{eE}{m\omega_c^2}\right) \\
&= -\frac{ev_d}{L_y}
\end{aligned}
$$

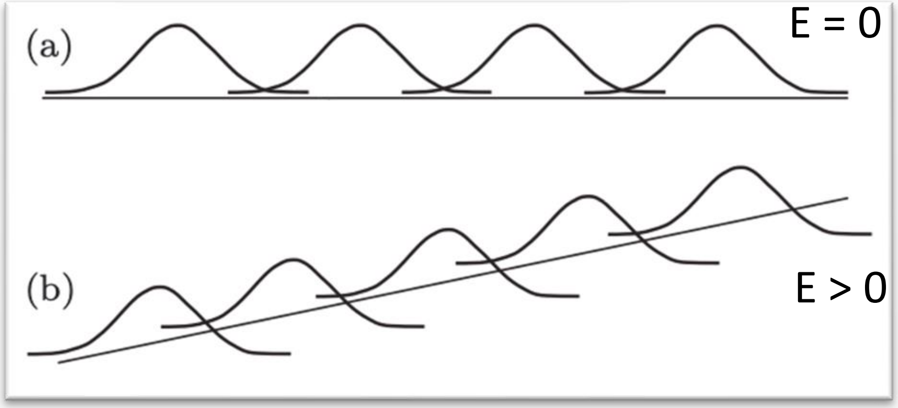
Current density

$$
\begin{aligned}
j_y &= \frac{L_y}{L_x} \int_0^{L_x/l_B^2} \frac{\mathrm{d}k}{2\pi} I_y(k) \\
&= -ev_d \frac{1}{L_x} \frac{L_x}{2\pi l_B^2} \\
&= -\frac{e^2}{h} E
\end{aligned}
$$

Hall conductivity

$$\sigma_{yx} = -\frac{e^2}{h}$$

Current response



$$\varepsilon_{nk} = (n + \tfrac{1}{2})\hbar\omega_c + eEx'_k + \frac{1}{2}mv_{\mathrm{d}}^2$$

Electric field lifts degeneracy within a LL, leading to dispersion and the drift velocity

Quantized Hall conductivity $\sigma_{yx} = -\frac{e^2}{h}$

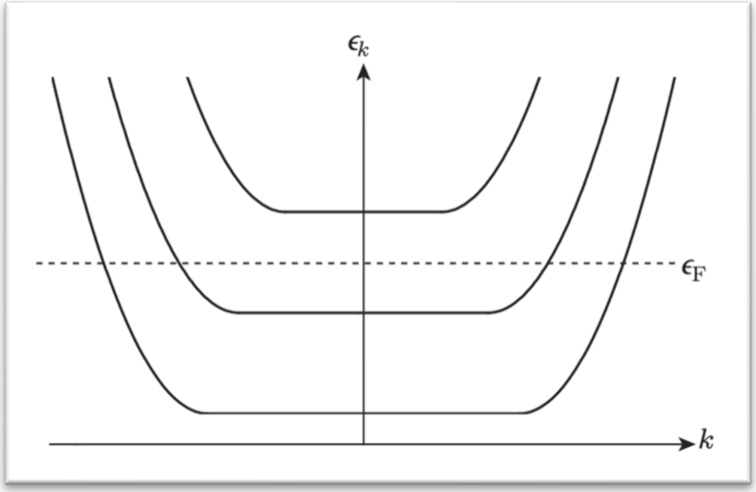
It is unclear:

- how does an filled band (insulator) carry (Hall) current?
- why the Hall conductivity is quantized?

Edge states

- Edges are modeled as confining potential

$$\varepsilon_{nk} \approx (n + \tfrac{1}{2})\hbar\omega_c + V(x'_k) + \cdots$$



- Group velocity

$$\boldsymbol{v}_k = \frac{1}{\hbar}\frac{\partial \varepsilon_{kn}}{\partial k}\hat{y} \approx \frac{l_B^2}{\hbar}\frac{dV(x)}{dx}\bigg|_{x=X_k}\hat{y}$$

This leads to unidirectional motion of electrons on the edges: chiral edge states

Hall current carried by the edge

We use the Landauer formula

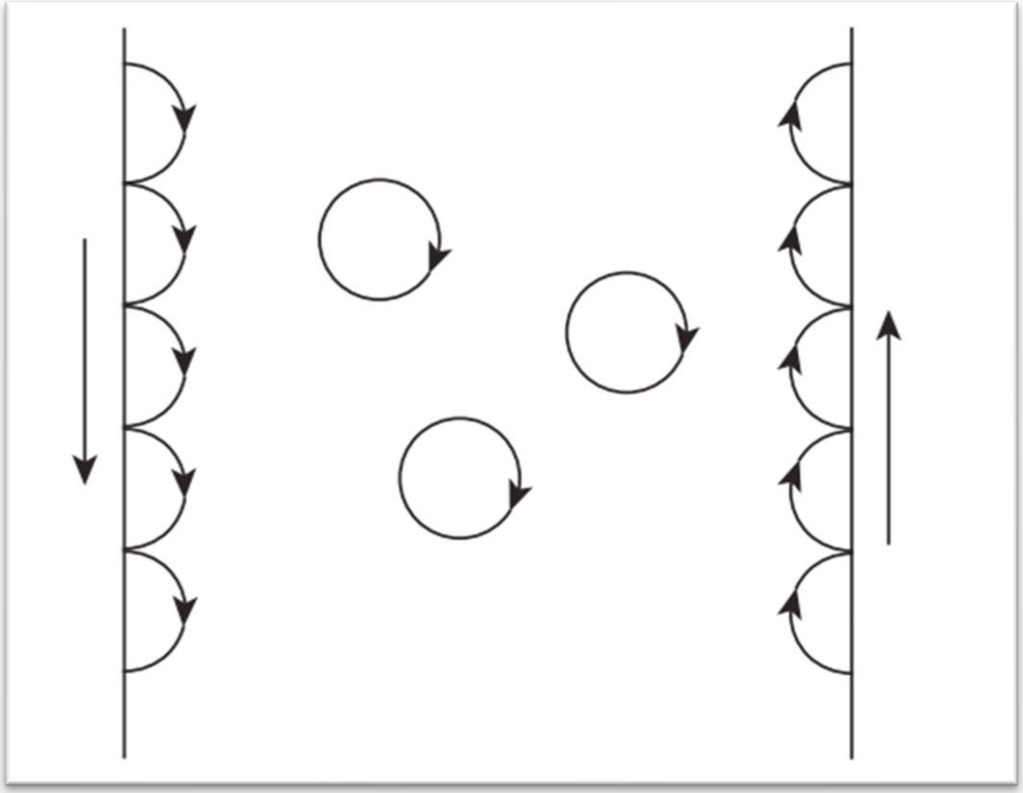
$$\begin{aligned} I &= -\frac{e}{L_y}\int_{-\infty}^{+\infty} dk \frac{L_y}{2\pi}\frac{1}{\hbar}\frac{\partial \epsilon_k}{\partial k} n_k \\ &= -\frac{e}{h}\int_{\mu_\mathrm{L}}^{\mu_\mathrm{R}} d\epsilon \\ &= -\frac{e}{h}\left[\mu_\mathrm{R} - \mu_\mathrm{L}\right] \end{aligned}$$

Hall voltage

$$(-e)V_\mathrm{H} \equiv (-e)[V_\mathrm{R} - V_\mathrm{L}] = [\mu_\mathrm{R} - \mu_\mathrm{L}].$$

$$I = \nu \frac{e^2}{h} V_\mathrm{H} \Rightarrow \begin{array}{l} \sigma_{xx} = 0 \\ \sigma_{xy} = \nu \dfrac{e^2}{h} \end{array}$$

Semiclassical picture of chiral edge states

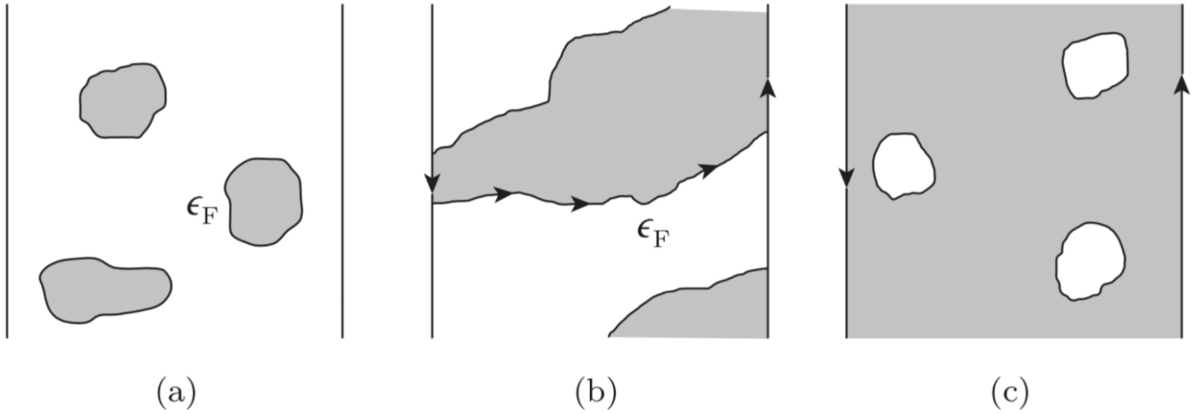


Skipping-orbit motion: no backscattering even with disorder

Semiclassical percolation

- Disorder in the bulk of 2DEG can be modeled as a random potential.
- Assume small hybridization between LLs

$$\mu_n^* = \left(n + \frac{1}{2}\right)\hbar\omega_c.$$



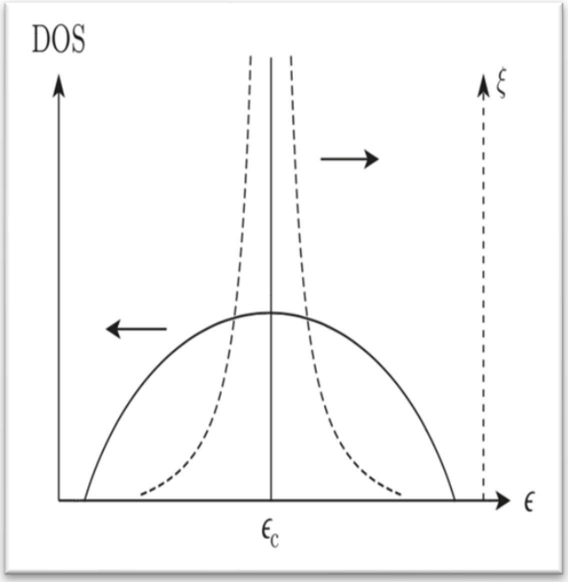
(a) (b) (c)

Insulating bulk
No electron on edges

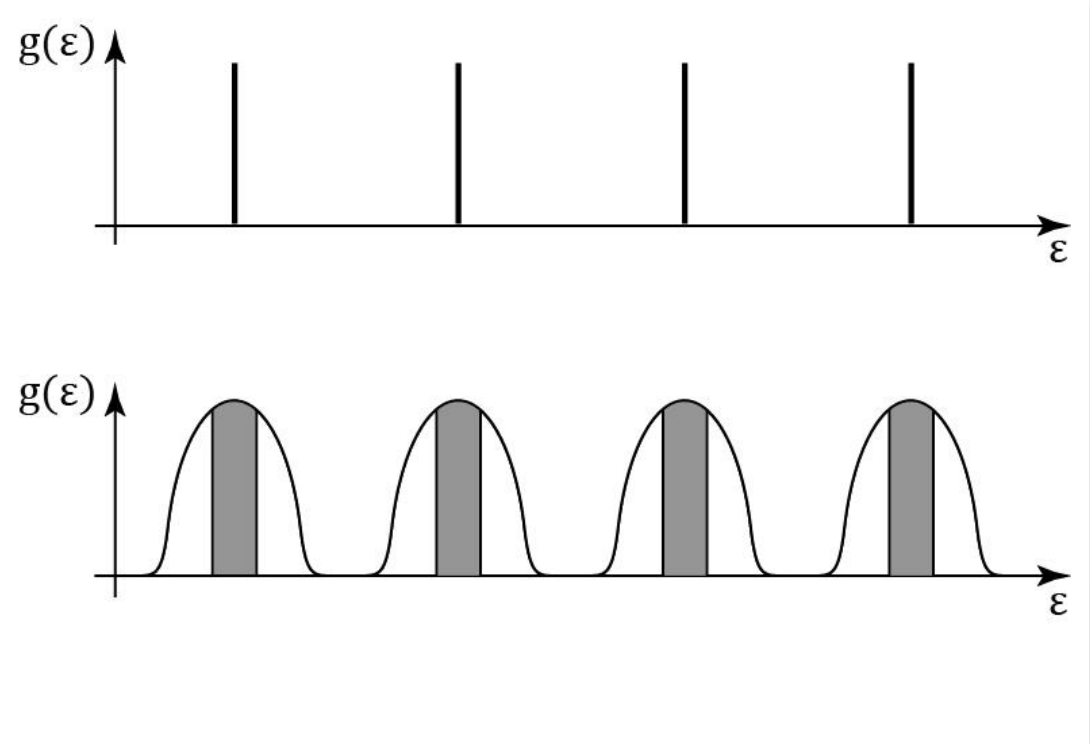
Insulating bulk
Chiral current on edges

Percolation transition

Percolation and localization



Broadened LLs with diorder



Density of states of (a) perfect quantum Hall system, and (b) with disorder. In (b), shaded regions are extended, and unshaded regions are localized.

Graphene in a uniform magnetic field

- The Hamiltonian $H_\tau = v(\tau\sigma_x p_x + \sigma_y p_y)$
- Uniforma B field $\boldsymbol{B} = -B\hat{z}$
- Peierls' substitution (minimal coupling) $\boldsymbol{\Pi} = \boldsymbol{p} + e\boldsymbol{a}$

Mechanical momentum:

$$H_\tau = v \begin{bmatrix} 0 & \tau\Pi_x - \mathrm{i}\Pi_y \\ \tau\Pi_x + \mathrm{i}\Pi_y & 0 \end{bmatrix}$$

$$\begin{aligned} [\Pi_x, \Pi_y] &= [-\mathrm{i}\hbar\partial_x + ea_x, -\mathrm{i}\hbar\partial_y + ea_y] \\ &= -\mathrm{i}\hbar([\partial_x, ea_y] + [\partial_y, ea_x]) \\ &= \mathrm{i}\hbar^2/l_B^2 \end{aligned}$$

Graphene in a uniform magnetic field

- Ladder operators $[a, a^\dagger] = 1$

$$a = \frac{l_B}{\sqrt{2}\hbar}\left(\Pi_x + i\Pi_y\right), \qquad a^\dagger = \frac{l_B}{\sqrt{2}\hbar}\left(\Pi_x - i\Pi_y\right)$$

- Then K, tau =1

$$H_K = v\left[\begin{array}{cc} 0 & \Pi_x - i\Pi_y \\ \Pi_x + i\Pi_y & 0 \end{array}\right] = \epsilon_{\rm D}\left[\begin{array}{cc} 0 & a^\dagger \\ a & 0 \end{array}\right]$$

$$\epsilon_{\rm D} = \sqrt{2}\hbar v_{\rm F}/l_B$$

Graphene in a uniform magnetic field

- Trick $H_K^2 = \epsilon_{\mathrm{D}}^2 \left[\begin{array}{cc} a^\dagger a & 0 \\ 0 & aa^\dagger \end{array} \right] = \epsilon_{\mathrm{D}}^2 \left[\begin{array}{cc} a^\dagger a & 0 \\ 0 & a^\dagger a + 1 \end{array} \right]$
- So H_K^2 is already diagonal, with eigenfuncion $[\phi_n, \phi_{n-1}]^T$
- Then the Landau levels are

$$\varepsilon_n = \pm\epsilon_{\mathrm{D}}\sqrt{|n|} = \mathrm{sgn}(n)\epsilon_{\mathrm{D}}\sqrt{|n|},$$
$$n = 0, \pm 1, \pm 2, \cdots$$

- The LL's are not equally spaced in energy
- Positive (electron) and negative (hole) LL's
- Zeroth LL, $\varepsilon_0 = 0$: anomalous quantum Hall sequence

Anomalous quantum Hall sequence of graphene

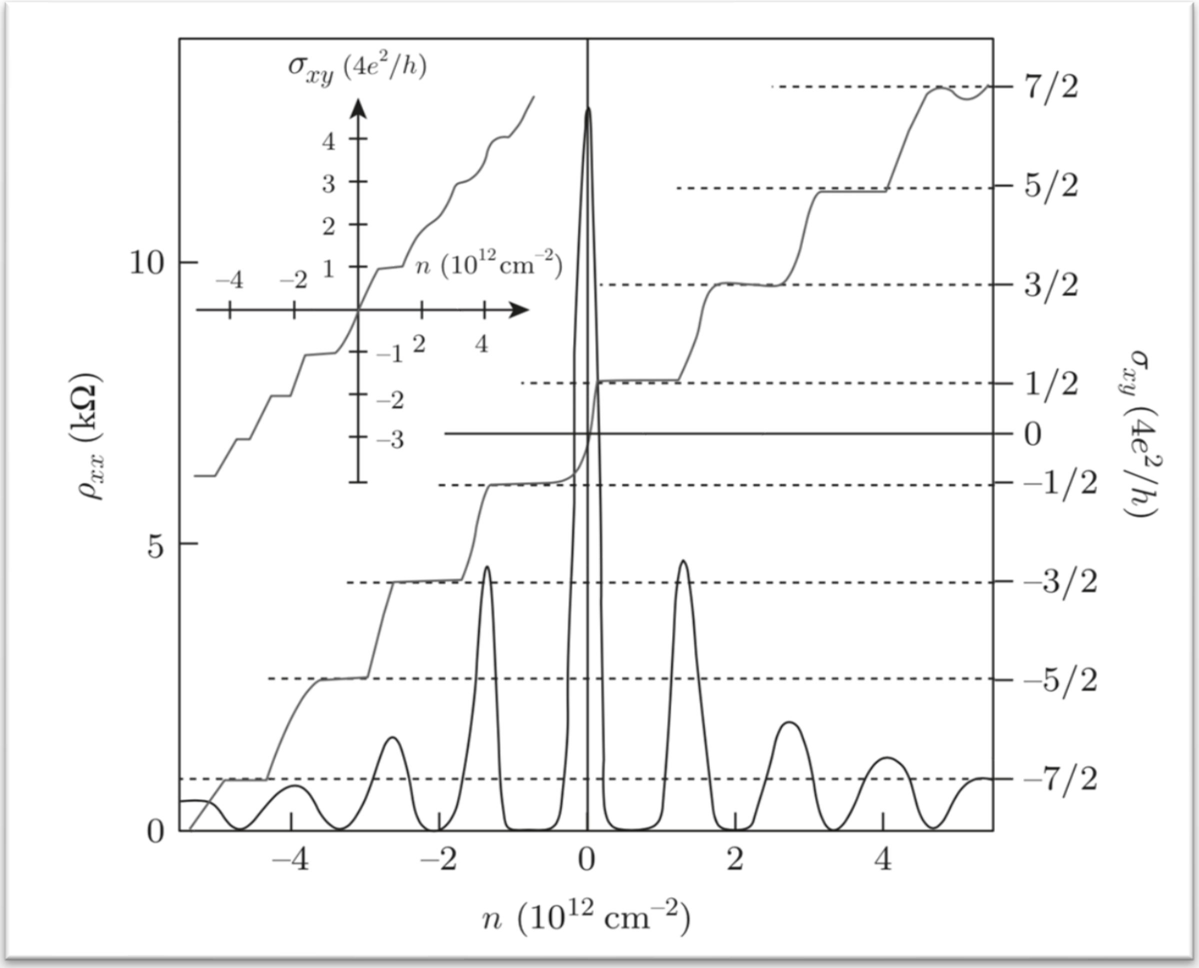
- A filled electron LL: +e^2/h
- A filled hole LL: -e^2/h
- at neutral point $\sigma_{xy} = 0$
- The zeroth LL:

K contributes (1/2) x e^2/h, K’ contributes (1/2) x e^2/h

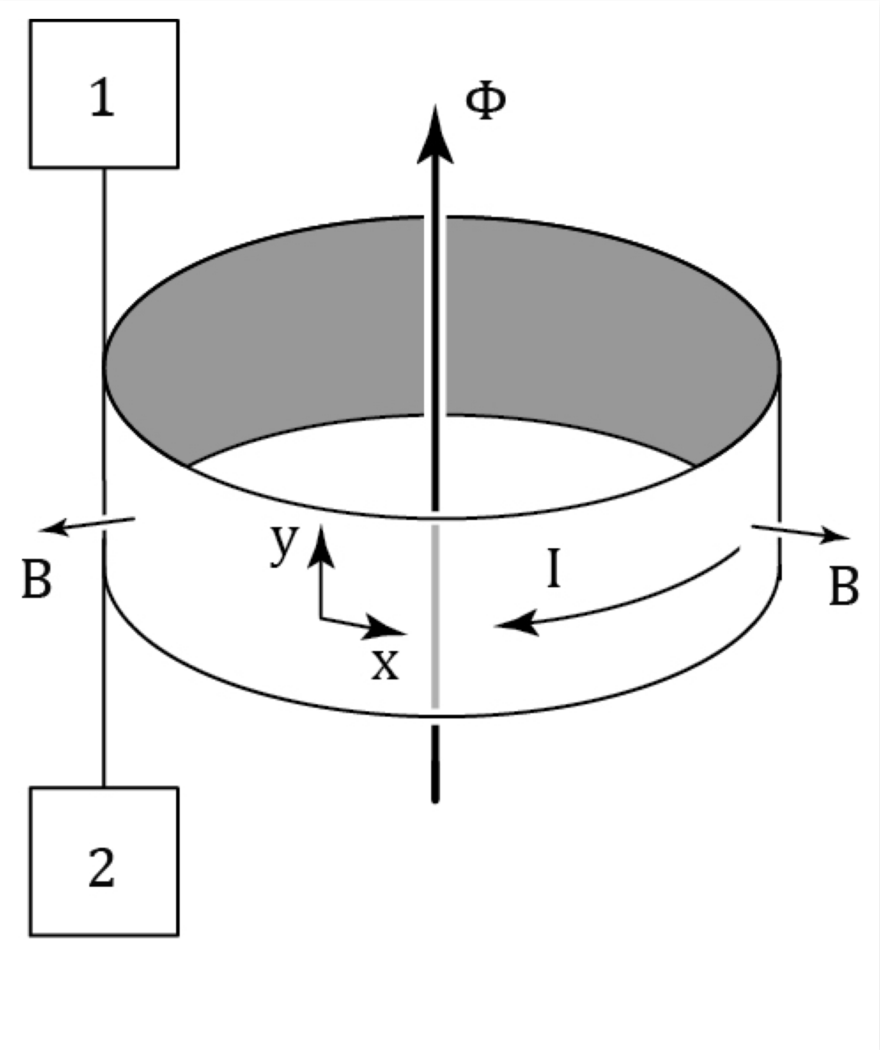
With spin degeneracy, the Hall step is 2 e^2/h

$$\sigma_{xy} = 4\left(n + \frac{1}{2}\right)\frac{e^2}{h} = (\pm 2, \pm 6, \pm 10, \ldots)\frac{e^2}{h}$$

IQHE of graphene



Laughlin's argument



Electromotive force $-\frac{\mathrm{d}\Phi}{\mathrm{d}t}$

Principle of gauge invariance: changing the flux by a quantum can only map the system into itself, or exchange it with an excited state.

Laughlin's argument

- If $\Delta\Phi = h/e$, then the electrons pumped from 1 to 2 must be an integer, *c*
- So the work done, as Φ increases by h/e, is

$$\Delta\Phi \times I = ce\Delta V$$

$$\sigma_{\mathrm{H}} = \frac{I}{\Delta V} = c\frac{e^2}{h}$$

- The problem with Laughlin's argument: c is (intrinsically) undetermined; it can be any integer.
- The integer is the Chern number (or TKNN number), which is $\pi(S^2)$. We need Brillouin zone!!

Magnetic Bloch theorem

- The Bloch theorem comes from lattice translation symmetry forming an Abelian group

$$T_R H T_R^{-1} = H \qquad\qquad [T_R, T_{R'}] = 0$$

- Uniform B-field on a 2DEG

$$H = \frac{1}{2m}(p + ea)^2 + v$$

- The vector potential is a linear function of coordinates r

$$a_\mu = F_{\mu\nu} r_\nu$$

$$B_\mu = \epsilon_{\mu\nu\gamma} \partial_\nu a_\gamma = \epsilon_{\mu\nu\gamma} F_{\gamma\nu}$$

Gauge transformation

- Under lattice translation

$$T_R H T_R^{-1} = \frac{1}{2m}(p_\mu + e a_\mu(r) + e F_{\mu\nu} R_\nu)^2 + V(r)$$

- So the magnetic Hamiltonian is not invariant under translation.
- But can we gauge out $F_{\mu\nu}R_\nu$?

$$\nabla\chi = F_{r\nu}R_v$$

$$\chi = r_\mu F_{\mu\nu} R_\nu$$

$$e^{ie\chi/\hbar} T_R H T_R^{-1} e^{-ie\chi/\hbar} = H$$

- That is, H is invariant after a translation and a gauge transformation

Magnetic translation

- Introduce a magnetic translation $M_R = e^{\mathrm{i}e\chi/\hbar} T_R$
- Acting on a function

$$M_R \psi(\boldsymbol{r}) = \exp\left(\mathrm{i}\frac{e}{\hbar} F_{\mu} r_{\mu} R_{\nu}\right) \psi(\boldsymbol{r} + \boldsymbol{R})$$

- Acting twice

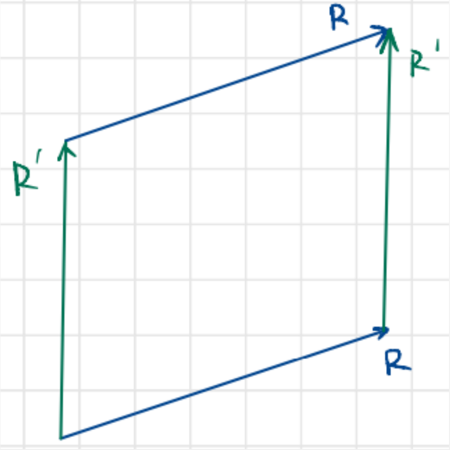
$$M_{R'} M_R \psi(\boldsymbol{r}) = \exp\left(\frac{ie}{\hbar} F_{r\eta} r_{\gamma} R'_{\eta}\right) \exp\left(\frac{ie}{\hbar} F_{\mu\nu} (r_r + R'_r) R_v\right) \psi(\boldsymbol{r} + \boldsymbol{R} + \boldsymbol{R}')$$

$$= \exp\left(\frac{\mathrm{i}e}{\hbar} F_{rv} R'_r R_v\right) \exp\left(\frac{\mathrm{i}e}{\hbar} F_{\gamma\eta} r_{\gamma} (R'_{\eta} + R_{\eta})\right) \psi(\boldsymbol{r} + \boldsymbol{R} + \boldsymbol{R}')$$

Magnetic translation

$$\begin{aligned} M_{R'} M_R &= \exp\left(\frac{\mathrm{i}e}{\hbar} F_{\mu\nu} R'_{\mu} R_{\nu}\right) M_{R+R'} \\ &= \exp\left(\frac{\mathrm{i}e}{\hbar} (F_{\mu\nu} - F_{\nu\mu}) R'_{\mu} R_{\nu}\right) M_R M_{R'} \end{aligned}$$

- So we have a magnetic translation operator that does leave Hamiltonian invariant, but M_R and $M_{R'}$ do not commute, in general
- But the phase is just the magnetic flux threading the parallelogram R’ x R



$$(F_{\mu\nu} - F_{\nu\mu}) R'_{r} R_{\nu} = \boldsymbol{B} \cdot \boldsymbol{R}' \times \boldsymbol{R} = \Phi$$

Magnetic translation

$$M_{R'}M_R = \exp(\mathrm{i}2\pi\Phi/\Phi_0)M_RM_{R'}$$

- No commutation means no Bloch theorem …
- Except when Φ/Φ_0 = integer !
- Let $B \cdot a_1 \times a_2 = \frac{p}{q}\Phi, \quad p, q \in \mathbb{N}$
- Consider the lattice vector $R_m = n_1 a_1 + n_2(q a_2)$
- Then for two consecutive translations commute for these

  magnetic translations

$$M_{R'_m}M_{R_m} = M_{R_m}M_{R'_m}$$

Magnetic Bloch theorem

$$M_{R_m}\psi_k(r) = e^{\mathrm{i}k\cdot R_m}\psi_k(r) \qquad\qquad R_m = n_1 a_1 + n_2(qa_2)$$

For a special subset of the full translation group of the crystal.

In group theory language, this is called a projective representation of the translation group.

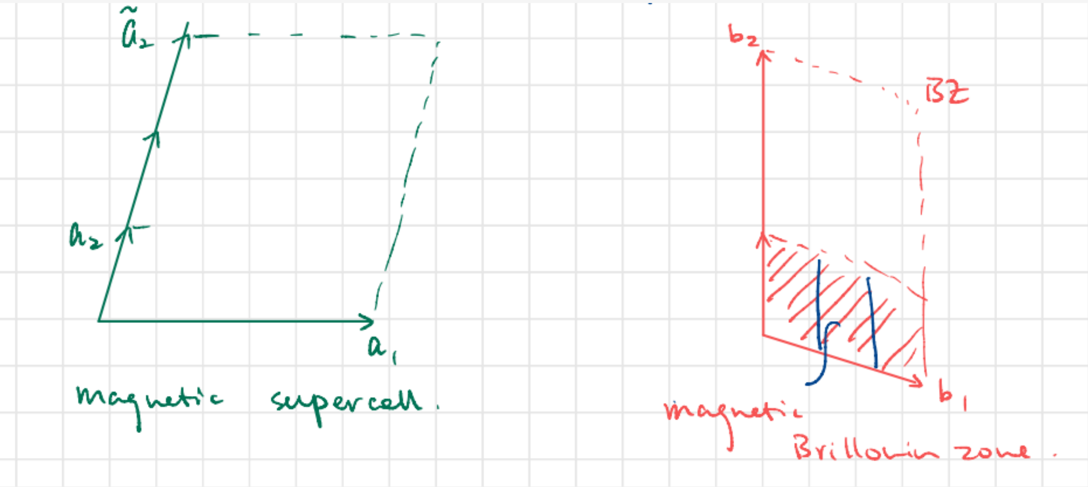
- Consider

$$\begin{aligned} M_{R_m}M_{a_2}\psi_k(r) &= \exp\left(\frac{ie}{\hbar}R_m \times a_2 \cdot B\right)M_{a_2}m_{R_m}\psi_k(r) \\ &= \exp[i\underbrace{\left(k + \frac{e}{\hbar}a_2 \times B\right)}_{k'} \cdot R_m]M_{a_2}\psi_k(r) \end{aligned}$$

Which means $M_{a_2}\psi_k$ is also a magnetic Bloch function at

Magnetic Brillouin zone

$$\begin{aligned} \boldsymbol{k}' = \boldsymbol{k} + \frac{e}{\hbar} a_2 \times B &= \boldsymbol{k} + \frac{e|B|}{\hbar} a_2 \times \hat{z} \\ &= \boldsymbol{k} + \frac{e|B|}{\hbar} \frac{a_1 \times a_2}{2\pi} \boldsymbol{b}_1 \\ &= \boldsymbol{k} + \frac{p}{q} \boldsymbol{b}_1 \end{aligned}$$



The magnetic Brillouin zone is composed of q degenerate pieces.