Formulas, constants and mathematical identities

Physics formulas

Berry connection

$$A_\alpha(\lambda) = \langle\psi(\lambda)|\mathrm{i}\frac{\partial}{\partial\lambda^\alpha}|\psi(\lambda)\rangle$$

Berry curvature

$$\Omega_{\alpha\beta} = \frac{\partial A_\beta}{\partial\lambda^\alpha} - \frac{\partial A_\alpha}{\partial\lambda^\beta}$$

$$d = 3 : \Omega_{\alpha\beta} = \epsilon_{\alpha\beta\gamma}\boldsymbol{\Omega}^\gamma$$

Phase-space volume of a Bloch electron at quasimomentum $\boldsymbol{k}$ in a magnetic field $\boldsymbol{B}$

$$\left(1 + \frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}(\boldsymbol{k})\right)\frac{\mathrm{d}^d r \mathrm{d}^d k}{(2\pi)^d}, \text{ where } \boldsymbol{\Omega}^\gamma(\boldsymbol{k}) = \frac{1}{2}\epsilon^{\gamma\alpha\beta}\Omega_{\alpha\beta}(\boldsymbol{k})$$

Einstein relation

$$\sigma = e^2\frac{\partial n}{\partial\mu}D$$

Density n of $T = 0$ d-dimensional free electron gas (d维自由电子气):

$$\alpha(d)n = k_F^d, \text{ where } \alpha(1) = \pi/2, \alpha(2) = 2\pi, \alpha(3) = 3\pi^2.$$

Pauli matrices

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}; \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Fourier transform

$$\begin{aligned} f(\boldsymbol{k}) &= \int \mathrm{d}^d r e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{r}) \\ f(\boldsymbol{r}) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{k}) \end{aligned}$$

$$\int \mathrm{d}^d r e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = (2\pi)^d\delta(\boldsymbol{k})$$

Taylor Series

$$\begin{aligned} \sqrt{1+x^2} &= 1 + \frac{x^2}{2} + O(x^4); \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + O\left(x^4\right); \\ \log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + O\left(x^4\right). \end{aligned}$$

Solid state theory 2023

Midterm exam

Closed book. No electronic devices allowed, except for a hand-held calculator.
April 17, Monday. 10:10 am –12:00 pm
503 Instruction Building No. 3. 三教503

1. Short questions (40 points)

(1) For a metal with a single band, the anomalous Hall conductivity is

$$\sigma_{xy} = -\frac{e^2}{\hbar}\int \frac{\mathrm{d}^d k}{(2\pi)^d} f_0(\varepsilon_{\boldsymbol{k}})\Omega(\boldsymbol{k}), \tag{1}$$

where $f(\varepsilon)$ is the Fermi-Dirac distribution for a given chemical potential μ, and $\Omega(\boldsymbol{k}) = \Omega_{xy}(\boldsymbol{k})$ is the Berry curvature of the Bloch state at $\boldsymbol{k}$. Please derive (推导) the Strèda formula for the Hall conductivity

$$\sigma_{xy} = \lim_{B\to 0}\left[-e\left(\frac{\partial n}{\partial B}\right)_\mu\right], \tag{2}$$

by considering a uniform, static and weak magnetic field (均匀静态弱磁场) $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. Note: do not use a time-dependent magnetic field (不要使用变化的磁场).

(2) A 2-dimensional Dirac Fermion which has two energy bands (能带)

$$\varepsilon_s(k) = s\hbar v k \tag{3}$$

in which $s = \pm 1$ is the band index (能带指标), $v > 0$ is the Fermi velocity, and $\boldsymbol{k} = (k_x, k_y)$ is the 2-dimensional wavevector (波矢). Please sketch the particle-hole continuum of the non-interacting density response function $\chi_0(q,\omega)$, at zero temperature for a Fermi energy $\varepsilon_F = \hbar v k_F > 0$.

(3) Give a physical picture (either in real space or $\boldsymbol{k}$-space 实空间或者倒空间) for weak localization for an electron in a metallic system *with time-reversal symmetry* (具有时间反演对称) with many impurities (杂质).

2. Plasmon dispersion of 2D electron gas (30 points)

(1) The susceptibility of a free electron gas (自由电子气) is given by

$$\chi_0(q,\omega) = 2\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{f_{\boldsymbol{k}} - f_{\boldsymbol{k}+\boldsymbol{q}}}{\hbar\omega + \varepsilon_{\boldsymbol{k}} - \varepsilon_{\boldsymbol{k}+\boldsymbol{q}} + \mathrm{i}0^+}, \tag{4}$$

where $\boldsymbol{q} = (q_x, q_y)$, $\boldsymbol{k} = (k_x, k_y)$. $f_{\boldsymbol{k}} = f(\varepsilon_{\boldsymbol{k}})$ is the Fermi-Dirac distribution. The dispersion relation for a 2-dimensional (2D) free electron gas is

$$\varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m},\ k = \sqrt{k_x^2 + k_y^2}. \tag{5}$$

Please verify that at $T = 0$ in the dynamical limit $q \to 0$ and $\omega > 0$

$$\chi_0(q, \omega) \approx \frac{nq^2}{m\omega^2} \left[1 + \frac{3}{4} \left(\frac{v_{\rm F} q}{\omega} \right)^2 \right], \tag{6}$$

where $v_{\rm F} = \hbar k_{\rm F}/m$.You will do so by performing the integal in Eq. (4) explicitly, by writing $\boldsymbol{k} \cdot \boldsymbol{q} = kq \cos\phi$, and by performing Taylor expansion in q of the denominator (分母). Probably you'd like to know

$$\int_0^{2\pi} (\cos\phi)^n \mathrm{d}\phi = 0 \text{ if } n \text{ is odd.} \tag{7}$$

(2) Recall the random-phase approximation (RPA)

$$\chi^{\rm RPA}(q, \omega) = \frac{\chi^0(q, \omega)}{1 - v_c(q)\chi^0(q, \omega)} \tag{8}$$

where

$$v_c(q) = \frac{e^2}{\epsilon_0 q} \tag{9}$$

is the 2D Coulomb potential (分母是q, 不是q^2), with ϵ_0 as the dielectric constant. Show tht the plasmon poles are given by

$$\Omega(q)^2 = \Omega_{\rm p}(q)^2 + \frac{3}{4} v_{\rm F}^2 q^2. \tag{10}$$

Give the form of $\Omega_p(q)$ in terms of constants m, e, n, ϵ_0.

3. Localization and scaling theory (30 points)

The weak-localization (弱局域化) correction (修正) to conductivity, $\delta\sigma$, of a 2-dimensional metal of size $L \times L$ is estimated as

$$\frac{\delta\sigma}{\sigma_0} = -\int_\tau^{\tau_\phi} \frac{\lambda v \mathrm{d}t}{\pi l_D(t)^2}. \tag{11}$$

in which τ is the transport relaxation time, and Drude conductivity is

$$\sigma_0 = \frac{ne^2\tau}{m}. \tag{12}$$

The diffusion length in time t is $l_D(t) = \sqrt{2Dt}$, where D is the diffusion coefficient. The dephasing time is taken to be the time for an electron to diffuse out of the sample $\tau_\phi = L^2/2D$. We assume the electron populates a single band with the dispersion

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m},$$

so that the Fermi wavelength $\lambda = 1/k_F$, and Fermi velocity $v = \hbar k_F/m$.

(1) Consider the the dimensionless conductance $g = \sigma/G_0$, where $G_0 = 2e^2/h$ is the conductance

quantum. According to the scaling hypothesis (标度假说)

$$\frac{\mathrm{d}\log g}{\mathrm{d}\log L} = \beta(g). \tag{13}$$

Show from the weak localization correction of conductivity that the scaling function $\beta(g)$ for large g (metallic regime) can be approximated as

$$\beta(g) \approx -\frac{a}{g}, \tag{14}$$

where a is a positive $O(1)$ constant. Give an explicit form of a. Sketch a curve of $\beta(g)$ vs $\log g$, and indicate renormalization flow directions. And describe briefly what is going to happen at large and small g limits according to the scaling law (13).

(2) In the presence of spin-orbit scattering which can flip (翻转) spins, we can introduce a spin damping (阻尼) term to the weak localization correction,

$$\frac{\delta\sigma}{\sigma_0} = -\int_{\tau}^{\tau_\phi} \frac{\lambda v \mathrm{d}t}{\pi l_D(t)^2} \times \left(\tfrac{3}{2}e^{-t/\tau_{\mathrm{so}}} - \tfrac{1}{2}\right). \tag{15}$$

For short times $t \ll \tau_{\mathrm{so}}$, spin flip is insignificant, we expect weak localization; for long times $t \gg \tau_{\mathrm{so}}$, spins are flipped and weak anti-localization (弱反局域化) occurs. With a reasonable approximation (引入合理近似), demonstrate that for very fast spin-orbit scattering $\tau_{\mathrm{so}} \sim \tau \ll \tau_\phi$,

$$\beta(g) \approx +\frac{a/2}{g}, \tag{16}$$

for large g. Sketch a curve of $\beta(g)$ vs $\log g$, and indicate renormalization flow directions. And describe briefly what is going to happen at large and small g limits according to the scaling law (13).